Understanding quantum mechanics via information-theoretic reconstructions

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Key argument

Whatever else they may be:

- descriptions of independent reality [Bell, Ghirardi, Pearle, Valentini],
- relative states [Everett, Saunders, Wallace],
- subjective information states [Wheeler, Fuchs],

quantum states of systems are compendia of probabilities for the outcomes of possible operations we may perform on the systems:

“operational theories.”
Quantum mechanics will cease to look puzzling only when we will be able to derive the formalism of the theory from a set of simple physical assertions ("postulates", "principles") about the world. Therefore, we should not try to append a reasonable interpretation to the quantum mechanical formalism, but rather to derive the formalism from a set of experimentally motivated postulates.
Historical context

• Information-theoretic approach:

Quantum logical reconstruction
# Language

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<th>Formal representation</th>
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<td>System</td>
<td>Systems S, O, P…</td>
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<td>Information</td>
<td>Yes-no questions</td>
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<td>Fact (act of bringing about information)</td>
<td>Answer to a yes-no question (given at time $t$)</td>
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Cf. Beltrametti and Casinelli
Axioms

Axiom I: There is a maximum amount of relevant information that can be extracted from a system.

Axiom II: It is always possible to acquire new information about a system.
Old strategy: use Axiom I to derive lattice orthomodularity
Quantum logical reconstruction of the Hilbert space

1. Definition of the lattice of yes-no questions.
2. Definition of orthogonal complement.
3. Definition of relevance and proof of orthomodularity.
4. Introduction of the space structure.
5. Lemmas about properties of the space.
6. Definition of the numeric field.
7. Construction of the Hilbert space.
Consider $b$ such that it entails the negation of $a$: $b \rightarrow \neg a$

If the observer asks $a$ and obtains an answer to $a$…

…but then asks a genuine new question $b$, it means that the observer expects either a positive or a negative answer to $b$.

This, in turn, is only possible if information $a$ is no more relevant; indeed, otherwise the observer would be bound to always obtain the negative answer to $b$.

We say that, by asking $b$, the observer renders $a$ irrelevant.
Definition of Relevance

- Question $b$ is called irrelevant with respect to question $a$ if $b \land a^\perp \neq 0$.

- Trivial in Hilbert lattices: $x \leq y$ are relevant with respect to $y$, all others irrelevant.

- Non-trivial if used to derive what in Hilbert lattices is assumed.
Non-trivial notion of relevance

• Question $b$ is relevant with respect to question $a$

and

• $b \geq a$
Amount of information

• Assumptions:

  1. If relevance is not lost, the amount of information grows monotonously as new information comes in.

  2. The lattice contains all possible information (yes-no questions). Thus, there are sufficiently many questions as to bring about any \textit{a priori} allowed amount of information.
Proof of Orthomodularity

• By Axiom I there exists a finite upper bound of the amount of relevant information, call it $N$. Select an arbitrary question $a$ and consider a question $\tilde{a}$ such that \{a, $\tilde{a}$\} bring $N$ bits of information. Then $a^\perp \wedge \tilde{a} = 0$.

• Lemma: An orthocomplemented lattice is orthomodular if and only if $a \leq b$ and $a^\perp \wedge b = 0$ imply $a = b$.

• Question $b$ is relevant with respect to $a$; and question $\tilde{a}$ is relevant with respect to $b$.

• Consider \{a, b, $\tilde{a}$\}. If $b > a$, this sequence preserves relevance and brings about strictly more than $N$ bits of relevant information.

• From the contradiction follows $a = b$. 
Step 7: Construction of the Hilbert space

- **Theorem:**

  Let \( W(P) \) be an ensemble of yes-no questions that can be asked to a physical system and \( V \) a vector space over real or complex numbers or quaternions such that a lattice of its subspaces \( L \) is isomorphic to \( W(P) \).

  Then there exists an inner product \( f \) on \( V \) such that \( V \) together with \( f \) form a Hilbert space.
Kalmbach’s theorem

Infinite-dimensional Hilbert space characterization theorem:

Let $H$ be an infinite-dimensional vector space over real or complex numbers or quaternions. Let $L$ be a complete orthomodular lattice of subspaces of $H$ which satisfies:

(i) Every finite-dimensional subspace of $H$ belongs to $L$.

(ii) For every element $U$ of $L$ and for every finite-dimensional subspace $V$ of $H$, linear sum $U+V$ belongs to $L$.

Then there exists an inner product $f$ on $H$ such that $(H,f)$ is a Hilbert space with $L$ as its lattice of closed subspaces.
List of axioms

**Information-theoretic axioms:**

I. There is a maximum amount of relevant information that can be extracted from a system.
II. It is always possible to acquire new information about a system.
III. If information I about a system was obtained, then there is no further information J about the fact of bringing about information I.

**Supplementary assumptions:**

IV. For any two yes-no questions there exists a yes-no question to which the answer is positive if and only if the answer to at least one of the initial question is positive.
V. For any two yes-no questions there exists a yes-no question to which the answer is positive if and only if the answer to both initial questions is positive.
VI. The lattice of questions is complete.
VII. The underlying field of the space of the theory is one of the numeric fields R, C or D and the involutory anti-automorphism in this field is continuous.
Open questions

1. Meaning of the lattice structure.
2. Meaning of the numeric field.
3. Interpretation of probability.
5. Dimension of the Hilbert space.