Multipartite Entanglement in Quantum Information

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Talk Preview

I. Introduction

II. “Distance-like” Entanglement measures

III. Multiparty entanglement in QI

IV. LOCC access of information

V. Examples

VI. Conclusions

I. Introduction

Motivation

Use / Role of multiparty entanglement in Quantum Information

- Qualification and quantification of multiparty entanglement
- But, what can we do with this? – interpretation of multiparty entanglement measures
- Using multiparty entanglement in quantum information, in a way beyond bipartite entanglement

I. Introduction

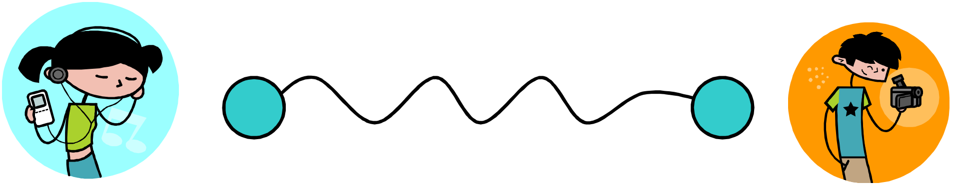
Two approaches to entanglement:

-Physical / operational

-Abstract

<u>Physical / Operational</u>

- Usefulness to quantum information tasks



*J. Eisert & D. Gross, quant-ph/0505149

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Bipartite case: unit of 'usefulness' is the singlet

$$|\varphi^{-}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

Entanglement of distillation	E_D	number of singlets we can distil from a state by LOCC
Entanglement of formation	E_F	number of singlets needed to create state LOCC

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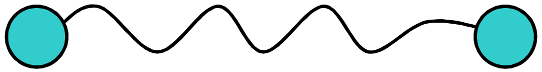

- Faithful Teleportation
- Secure Key Distribution

BOB

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Multiparty case:

….

PROBLEM! **– what are the units of usefulness?**

- inequivalent TYPES of entanglement: W, GHZ (no multiparty protocols..)

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I. Introduction

Abstract

- Satisfying axioms of entanglement - LOCC (Local Operations and Classical Communication) monotones*

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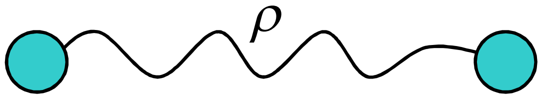

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$$\rho \xrightarrow{\text{LOCC}} \{\rho_i, p_i\}$$

$$E(\rho):\rho \to \mathrm{R}$$

$$E(\rho) \geq \sum_i p_i E(\rho_i)$$

Any function non-increasing under LOCC

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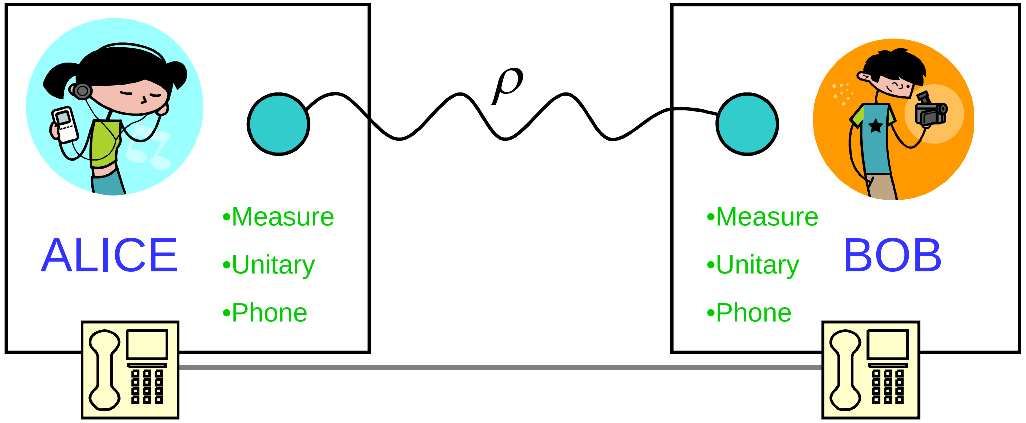

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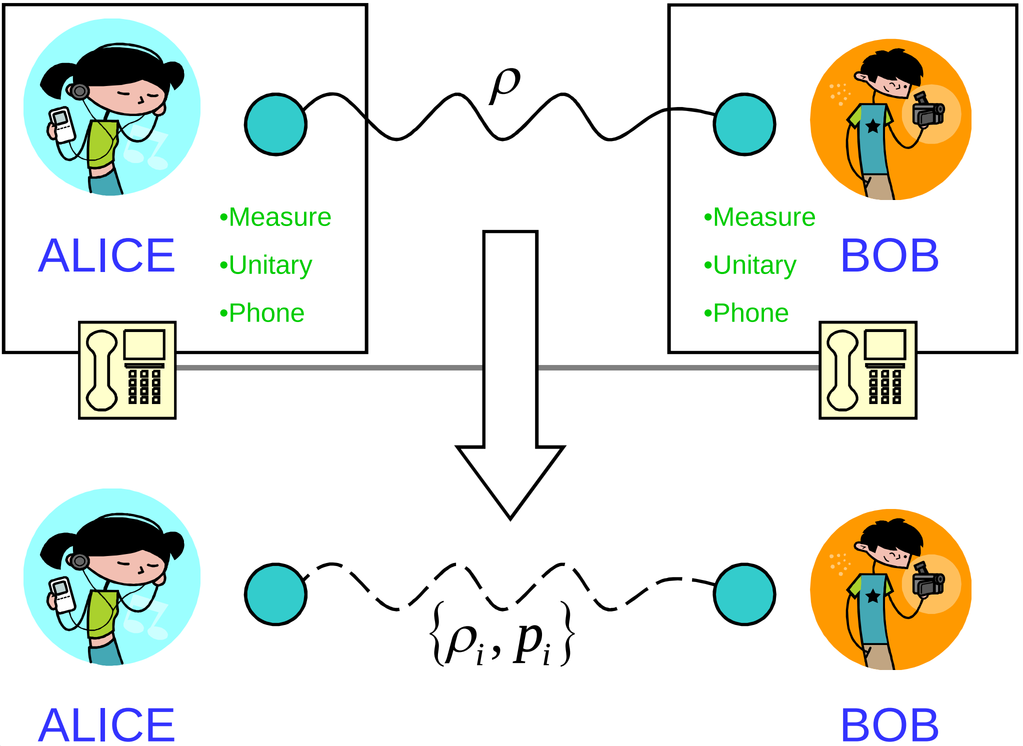

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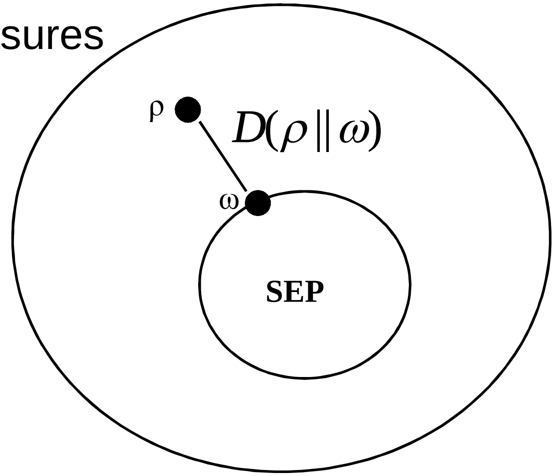
E.g. Distance like measures



given ρ , E(ρ) is the "distance" to the closest separable state ω

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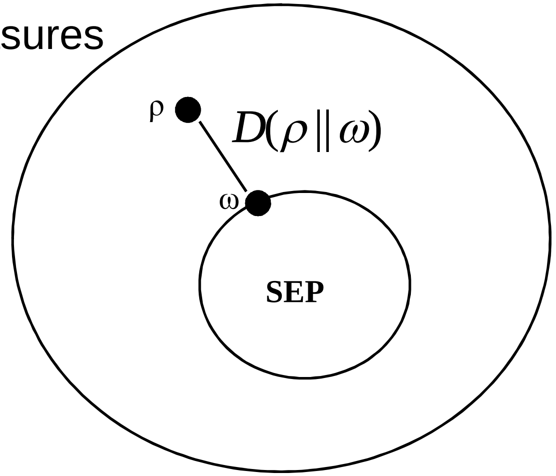


given ρ , E(ρ) is the "distance" to the closest separable state ω

$$E(\rho)=\min_{\omega} D(\rho\|\omega)$$

- Well defined for multipartite case!

But **- Difficult to calculate...**

- What do they mean?

*G. Vidal, J. Mod. Opt. 47(355), 2000.

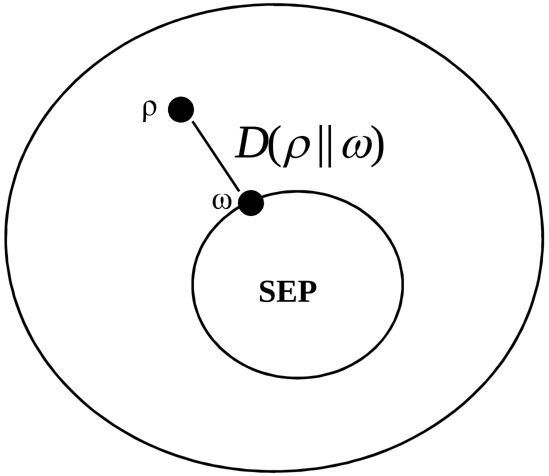
+ Wie, Ericsson, Goldbart, Munro, Quant Inf. Compu, 4, 252, (2004)

£Vidal, Tarrach, PRA 59, 141-155 (1999)

** Vedral , Plenio, PRA 57, 1619 (1998)

II. Entanglement

- For all states we have can place the distance-like entanglement measures in a hierarchy



$$r(\rho)\geq E_R(\rho)\geq E_G(\rho)$$

Geometric Measure

$$E_G(|\varphi\rangle):=-\log_2\max_{\phi\in prod}|\langle\varphi|\phi\rangle|^2$$

Relative entropy of entanglemet

$$E_R(\rho):=\min_{\omega\in sep}S(\rho\|\omega)$$

$$S(\rho\parallel\omega):=tr(\rho(\log_2\rho-\log_2\omega))$$

Logarithmic Robustness

$$\mathrm{r}(\rho):=\log(1+\min\ t)$$

$$\text{s.t.}\quad \omega=\tfrac{1}{(1+t)}(\rho+t\Delta)\in sep$$

II. Entanglement

$$r(\rho) \geq E_R(\rho) \geq E_G(\rho)$$

- When does equality hold?

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- take geometric closest state

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- if we choose U well, the bounds match and we get

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II. Entanglement

Twirling gives equality for

- Symmetric states (ground states for some many-body models, generalised W-states)

  - closest sep state has same symmetry, can easily find

$$|S(n,k)\rangle = \sum_{PERM} |00...01...11\rangle$$

$$E_G(|S(n,k)\rangle) = E_R(|S(n,k)\rangle) = r(|S(n,k)\rangle) = \log\left(\frac{\left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k}}{\left(\frac{n}{k}\right)} \right)$$

- 2 Colourable graph states (includes 1D,2D,3D Cluster, GHZ and Steane code states)

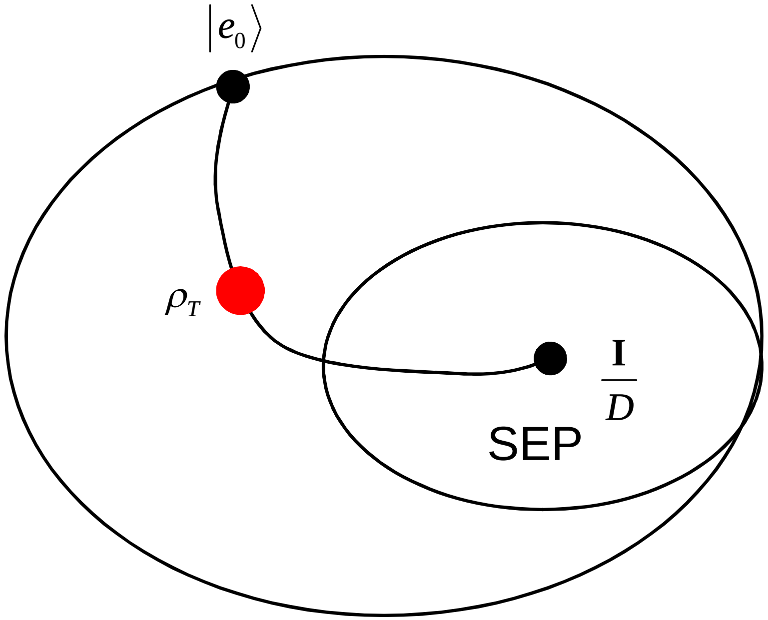
  - by means of the relationship to LOCC find upper and lower bounds which match

$$|G\rangle = \prod_{(j_1,j_2)\in Ed} CZ_{j_1,j_2} |+\rangle^{\otimes n}$$

$$E_G(|G\rangle) = E_R(|G\rangle) = r(|G\rangle) = \left\lfloor \frac{n}{2} \right\rfloor$$

II. Entanglement

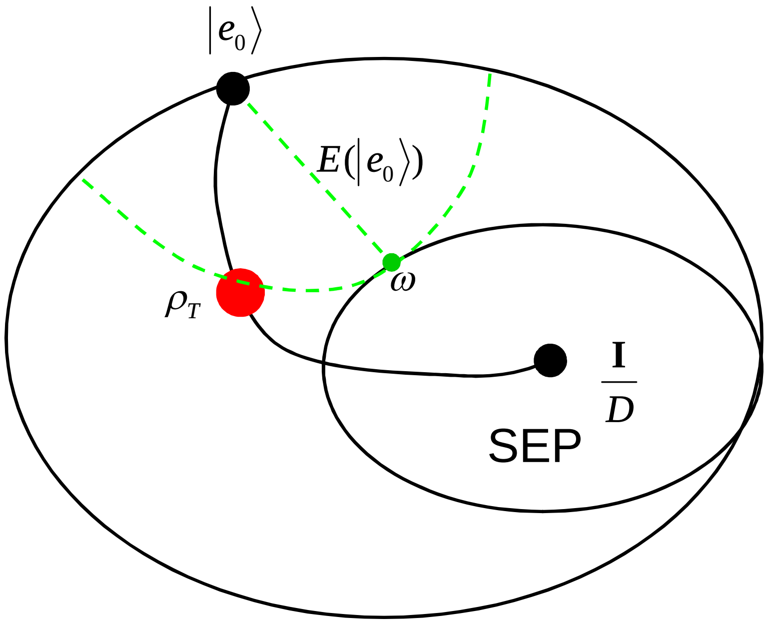
- Thermal states



$$\rho_T := \frac{1}{Z}\sum_i e^{-E_i/kT}|e_i\rangle\langle e_i|$$

II. Entanglement

- Witnessing entanglement in thermal states



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- If $D(|e_0\rangle \| \rho_T) < E(|e_0\rangle)$

$\Rightarrow$ ρ_T is entangled

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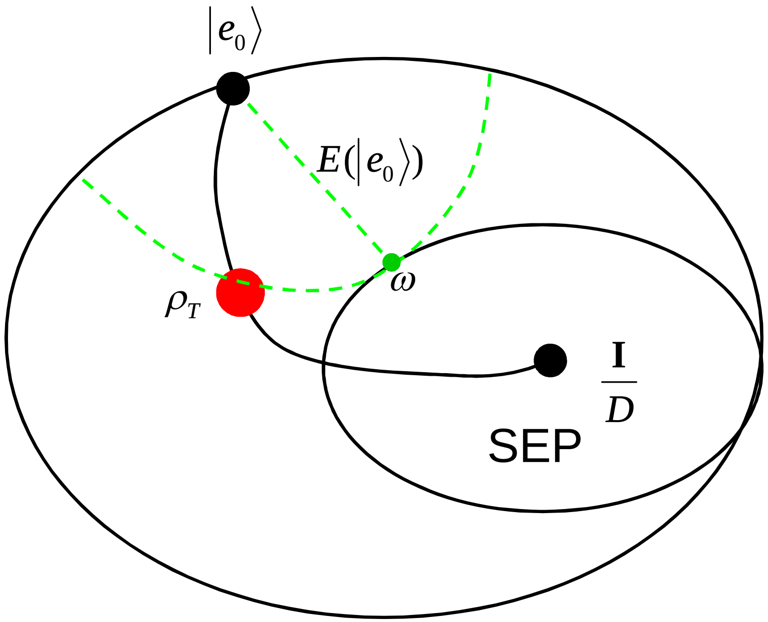

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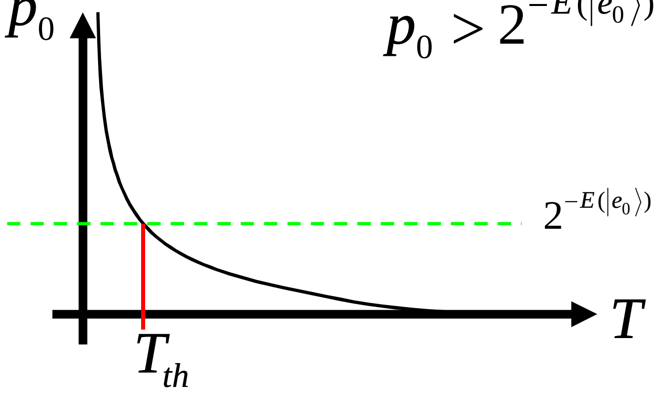
- *Witness of entanglement*

$$2^{-D(|e_0\rangle\|\rho_T)} = p_0 = \frac{e^{-E_0/kT}}{Z}$$

$$p_0 > 2^{-E(|e_0\rangle)}$$



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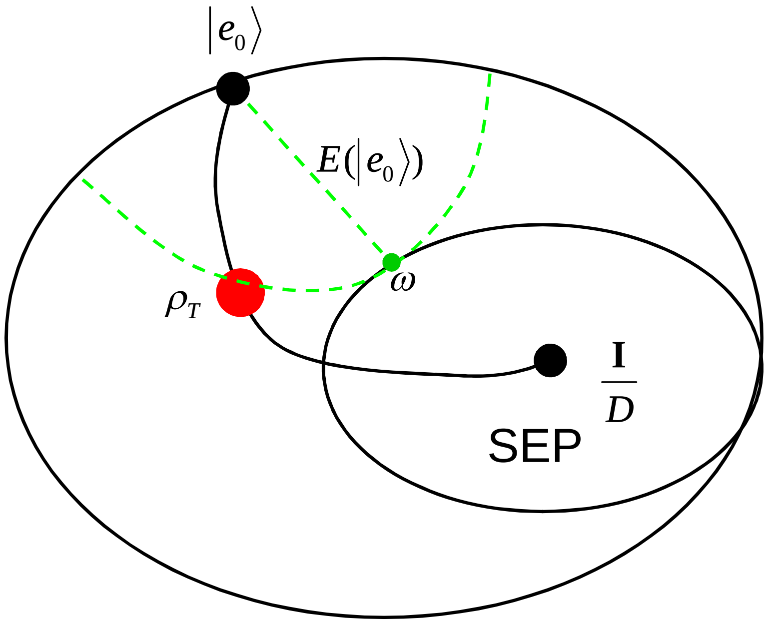

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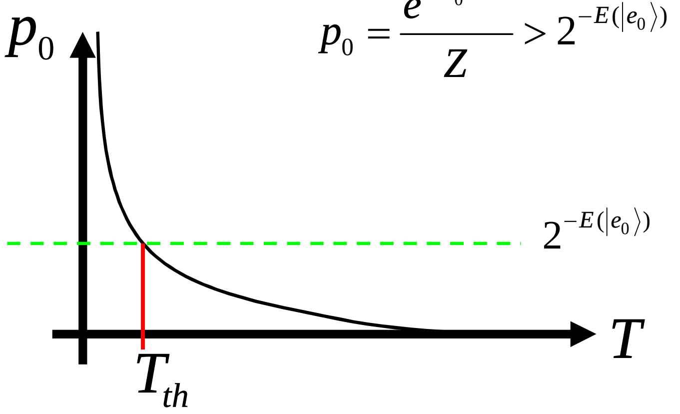
- Threshold temperature below which entanglement is guaranteed

- Critical point in terms of thermo quantities (free energy)

$$p_0 = \frac{e^{-E_0/kT}}{Z} > 2^{-E(|e_0\rangle)}$$



II. Entanglement

- Witnessing entanglement in thermal states

$$p_0 = \underbrace{\frac{e^{-E_0/kT}}{Z}}_{\text{Statistical Physics}} > 2^{-\underbrace{E(|e_0\rangle)}_{\text{Entanglement}}}$$

$$F = -kT \ln Z$$

$$e^{(F-E_0)/kT} > 2^{-E(|e_0\rangle)}$$

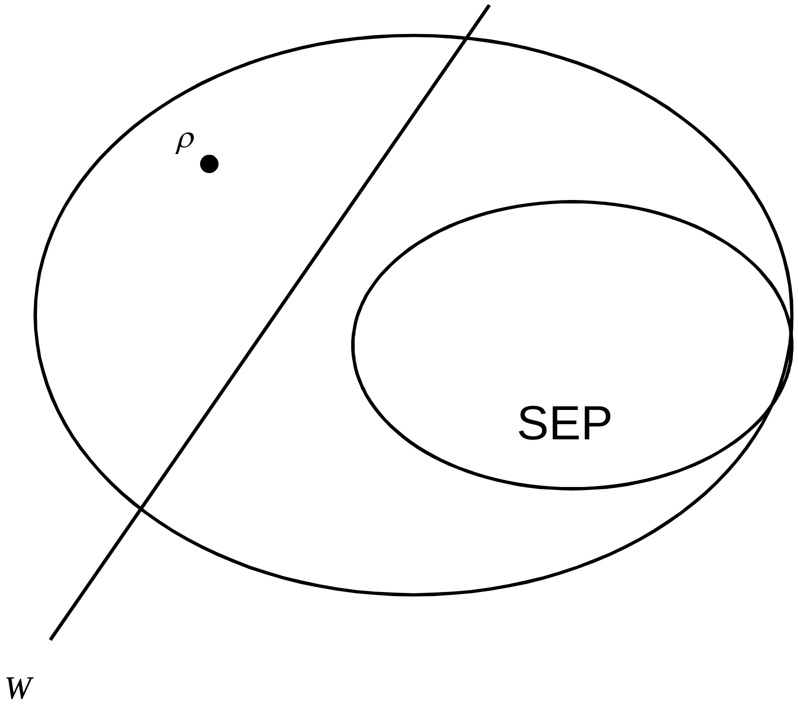
- Threshold temperature below which entanglement is guaranteed
- Critical point in terms of thermo quantities (free energy)
- Approximations also give valid witness

II. Entanglement

- Witnesses as bounds to entanglement

  - amount of violation gives a bound to the entanglement

Brandao PRA 05, Eisert et al quant-ph/0607167, Guhne et al quant-ph/0607163,



- Witness W hermitian operator such that

$$tr(\rho W) < 0 \quad \text{for some } \rho$$

$$\geq 0 \quad \text{for separable states}$$

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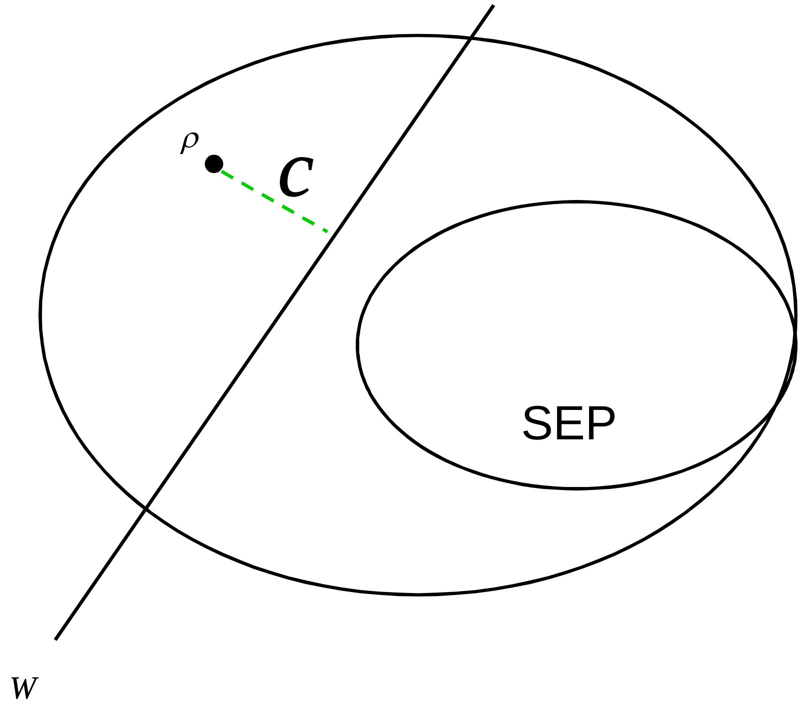

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$$E(\rho) \geq |c|$$

(plus some constants depending on measure)

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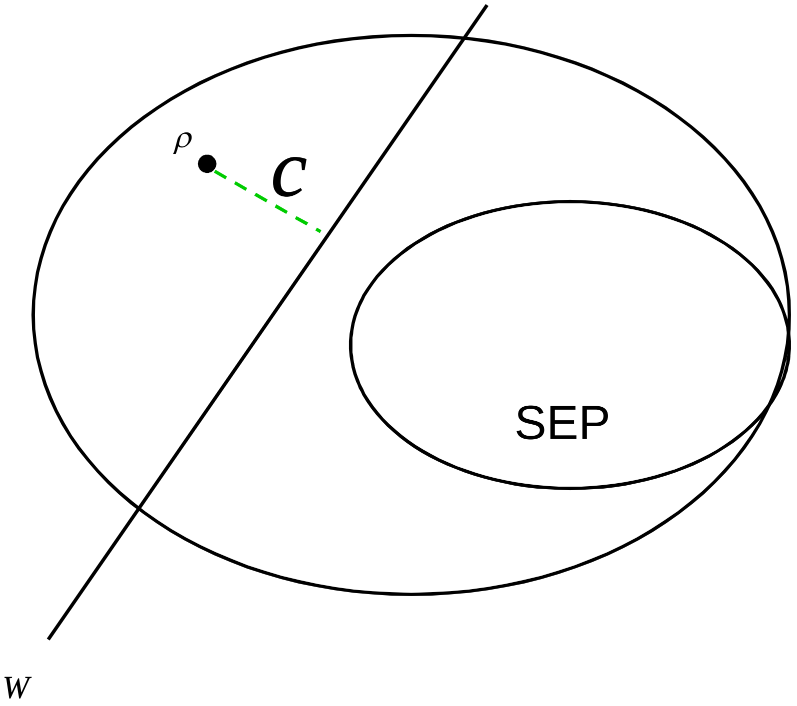

Experiment:

-Witness measured 4-photon Bourennane et al PRL 04

- gives exp. Entanglement bound Guhne et al quant-ph/06 E ≥ 0.209±0.023 (geometric meas.)

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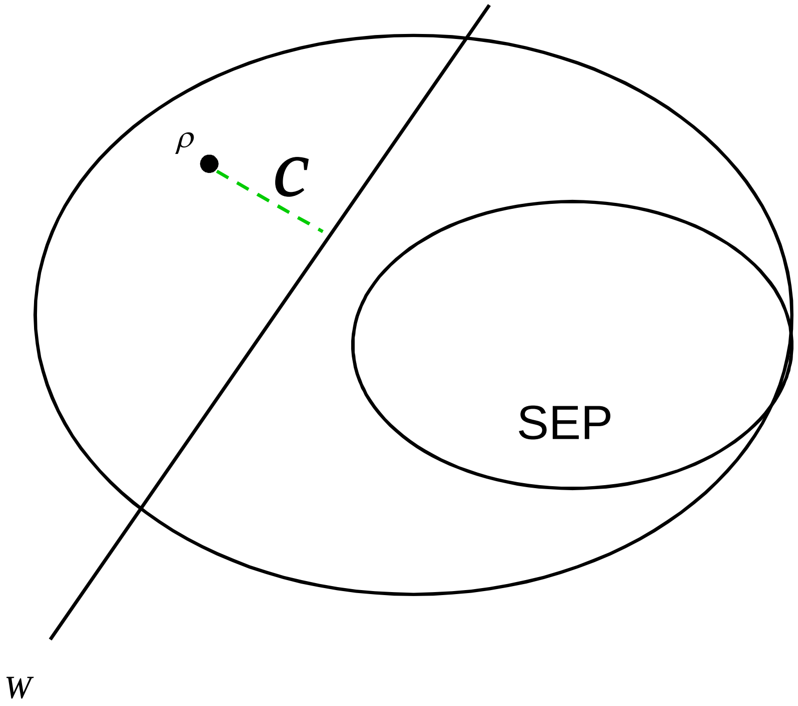

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Experiment:

-8-party entanglement witnessed in ion trap
Haeffner et al Nature 06

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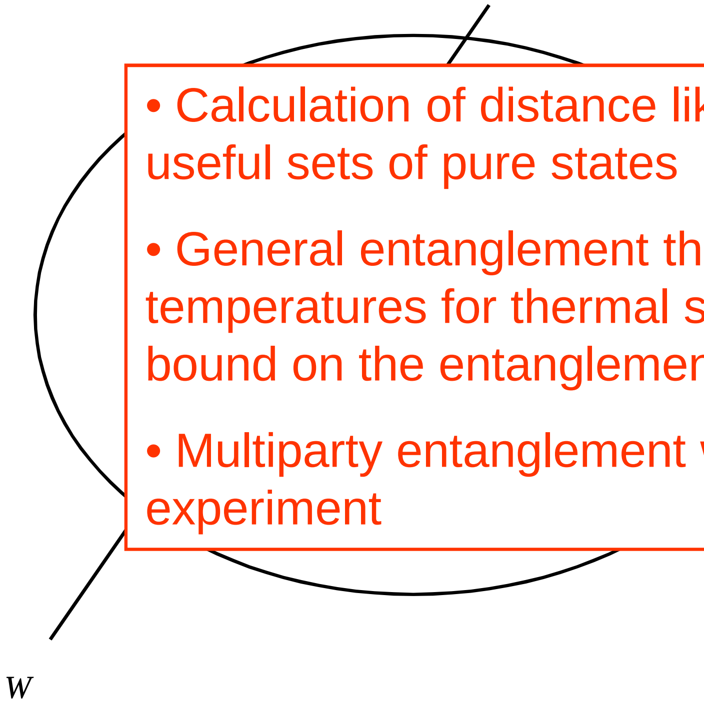

- Calculation of distance like measures for useful sets of pure states
- General entanglement threshold temperatures for thermal states (T as a bound on the entanglement)
- Multiparty entanglement witnessed in experiment

Experiment:

-Witness measured 4-photon Bourennane et al PRL 04

exp. Entanglement bound et al quant-ph/06 E ≥ 0.023 (geometric meas.)

< 0 for some ρ

≥ 0 for separable states

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Haeffner et al Nature 06

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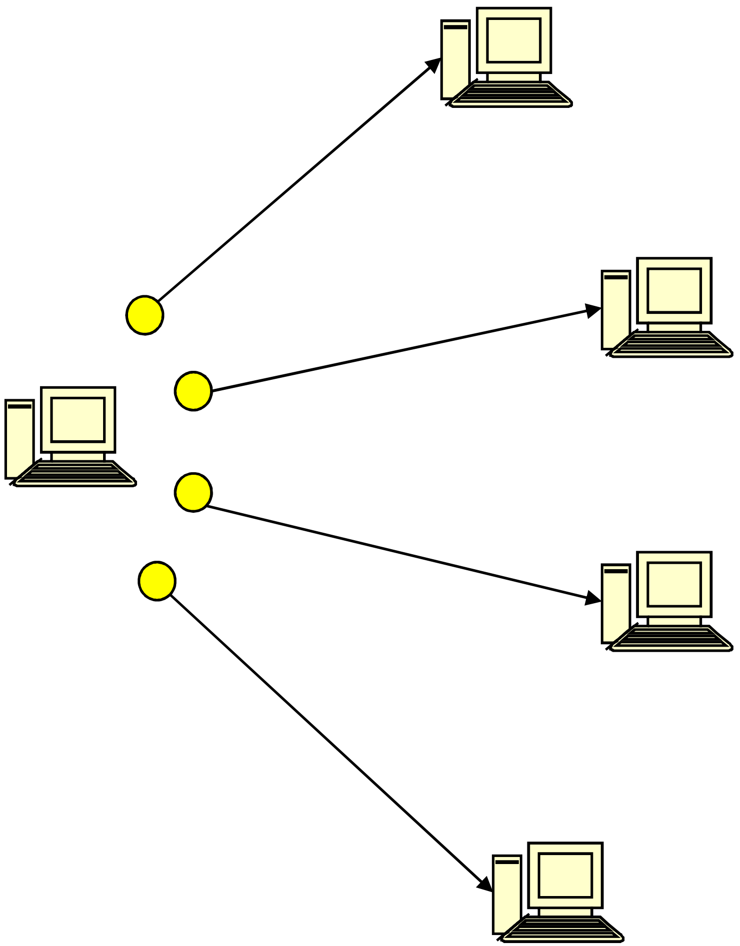
III. Multipartite Entanglement in QI

- Information distribution

  - Broadcast Channel



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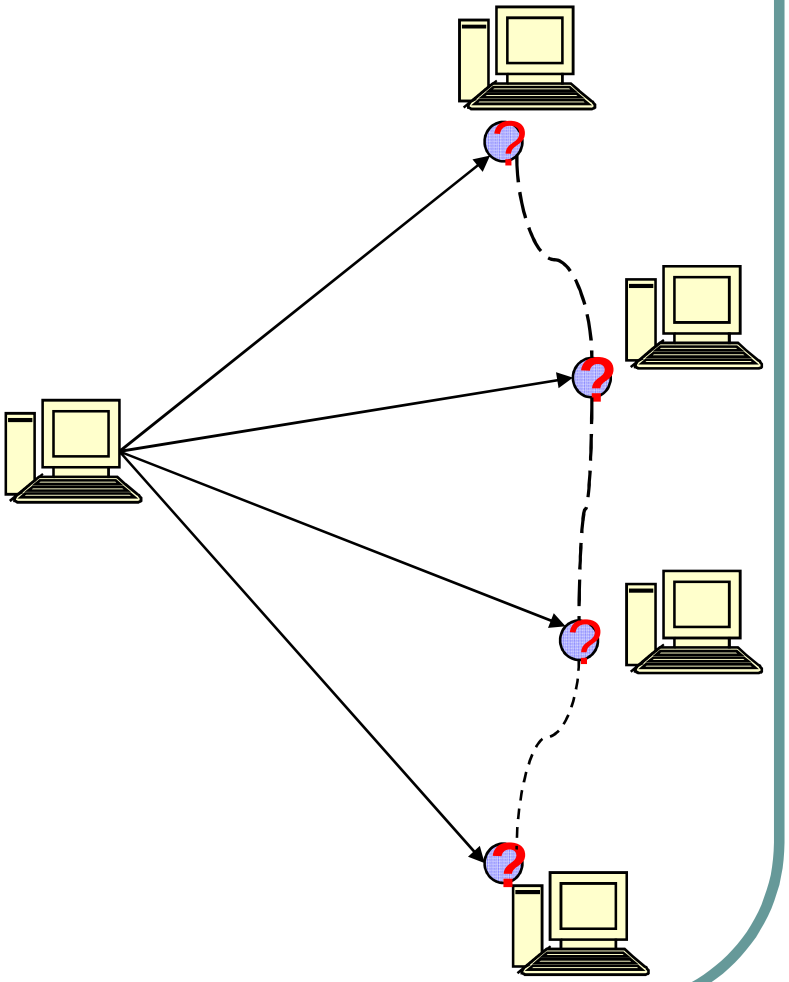
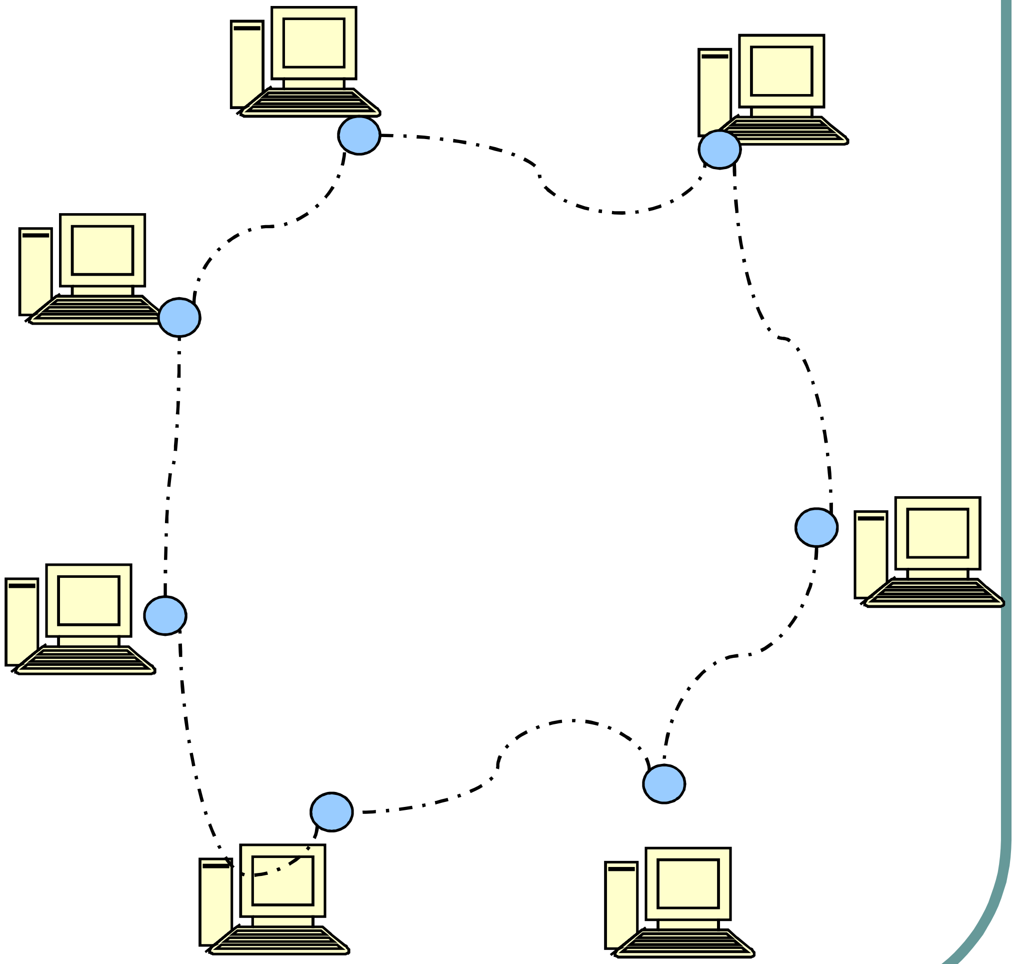
  - Secret sharing / data hiding



III. Multipartite Entanglement in QI

- Information distribution
  - - Broadcast Channel
  - - Secret sharing / data hiding

- Symmetry breaking
  - - Leader election, voting



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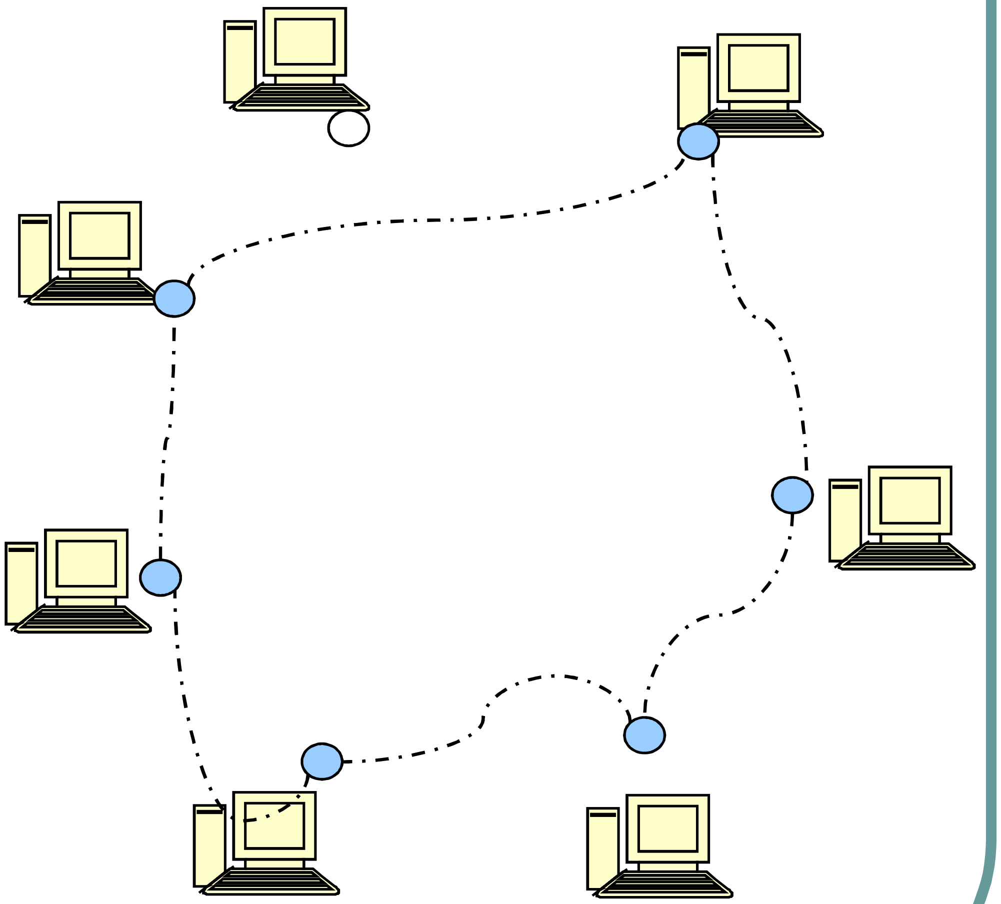

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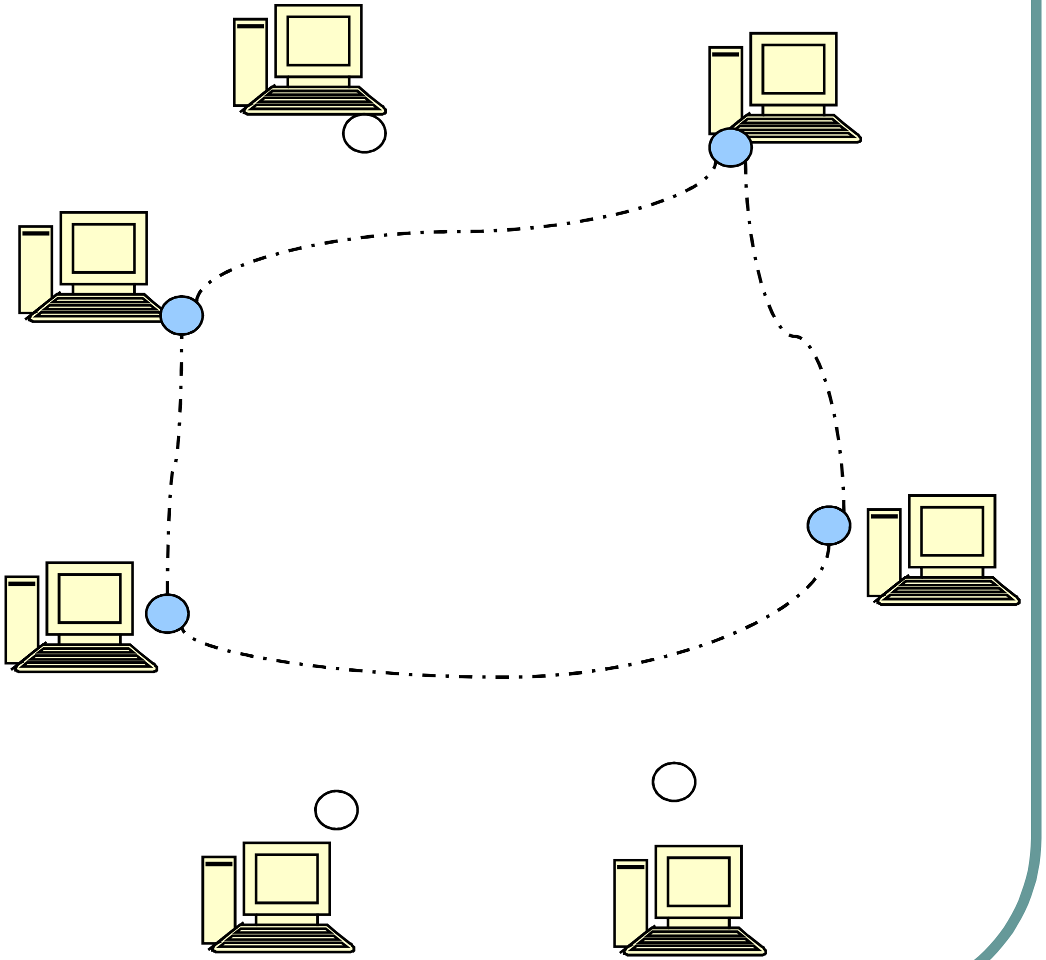

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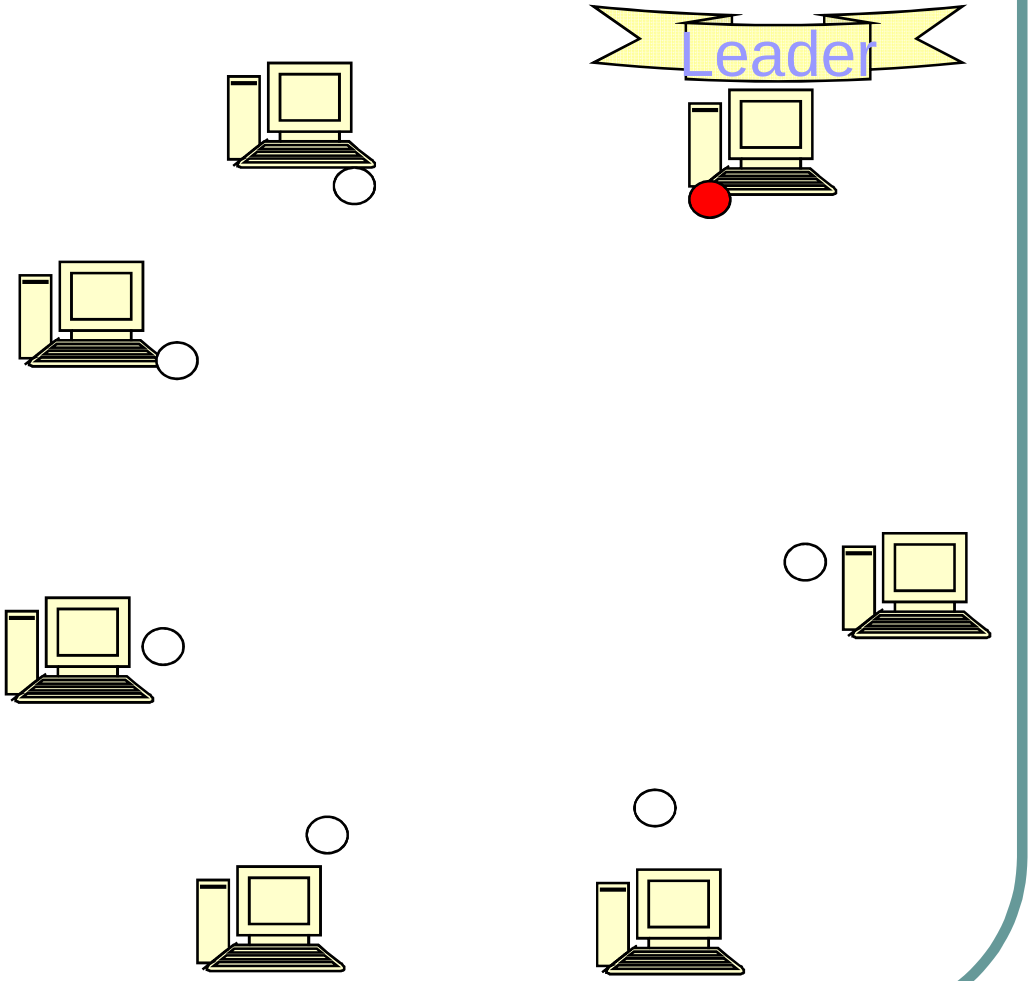

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  - Broadcast Channel
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- Symmetry breaking
  - Leader election, voting
- Quantum Computing
  - 1-way Q.C. / cluster state, application to Cryptography
  - role of entanglment in exponential speed up
  - distributed Q.C.

III. Multipartite Entanglement in QI

Experiment:

Single qubit secret sharing 6-party, Schmidt et al quant-ph/0502107

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Idea:

Only scratching the surface of possibilities, make use of richness of multiparty entanglement

- Begin by looking at LOCC access of information

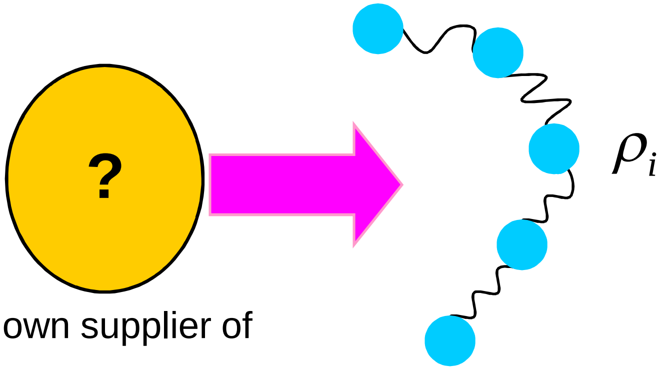
Experiment:

4-photon 1-way QC, Walter et al Nature 05

IV. LOCC access of information

The Problem

Classical information is encoded in states $\{\rho_i\}$ *,discover as well as possible which we are given, using only LOCC*



Some unknown supplier of states

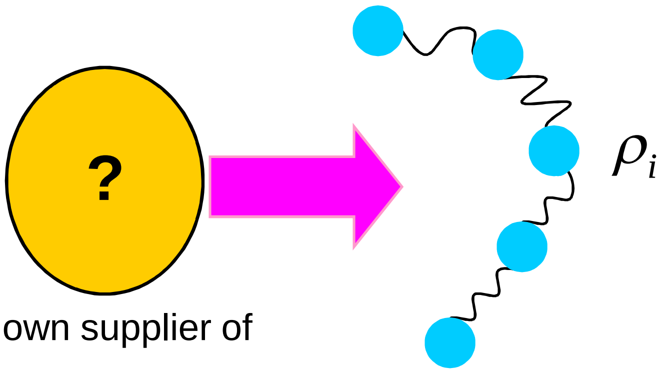
e.g. Oven, algorithm, Q Crypt, channel

- How well can we do this?
- Differences to global case?

IV. LOCC access of information

The Problem

Classical information is encoded in states $\{\rho_i\}$ *,discover as well as possible which we are given, using only LOCC*



Some unknown supplier of states

e.g. Oven, algorithm, Q Crypt, channel

- How well can we do this?
- Differences to global case?

➡ LOCC state discrimination

IV. LOCC access of information

Why access information by LOCC?

Could be many uses, here are a few

1) Environment(s) assisted channel capacity

2) Distribution of secret (secret sharing/ data hiding)

3) Network of Quantum Computers

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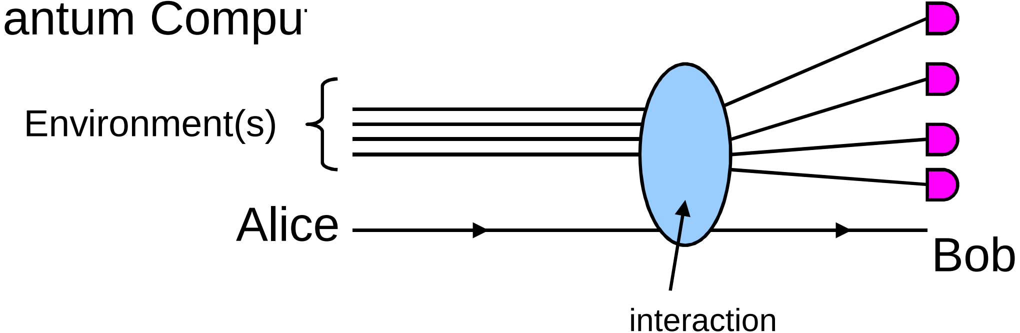
3) Network of Quantum Compu

-Channel gets entangled to the environment.

-Measure environment, and "correct", using LOCC



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* Hayden & King quant-ph/0409026, Winter quant-ph/0507045

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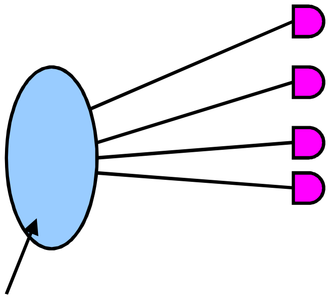
3) Netw

$C_{LOCC} < C_{Global}$

$$I_{secret} := I_{global} - I_{LOCC}$$



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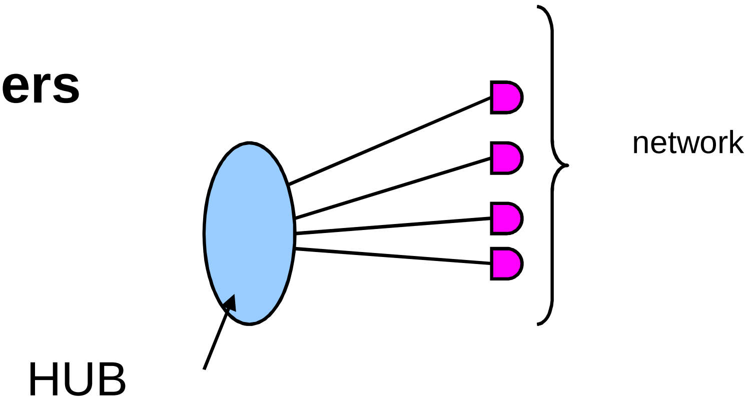

Results: entanglement hurts..

- **LOCC state discrimination:**
  Given a set of states, $\{\rho_i\}$ the number of states N we can discriminate perfectly by LOCC is bounded by

$$N \leq \frac{D}{2^{\overline{E(\rho_i)}}}$$

$E(\rho_i)$	Entanglement
D	total dimension
$\overline{x_i} := {}^{1}\!/_{N} \sum_i^N x_i$	average

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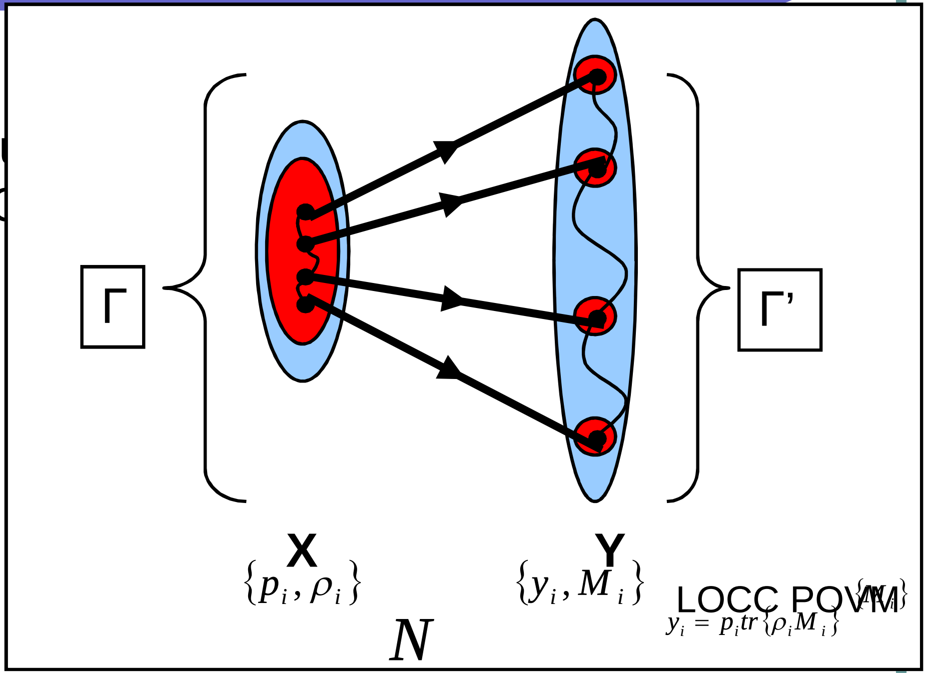

- **LOCC Channel Capacity:**
  Encoding into the fixed letter states, $\{\rho_i\}$ the LOCC decoded channel capacity is bounded by

$$C \le \log_2 D - E_{\min}(\rho_i)$$

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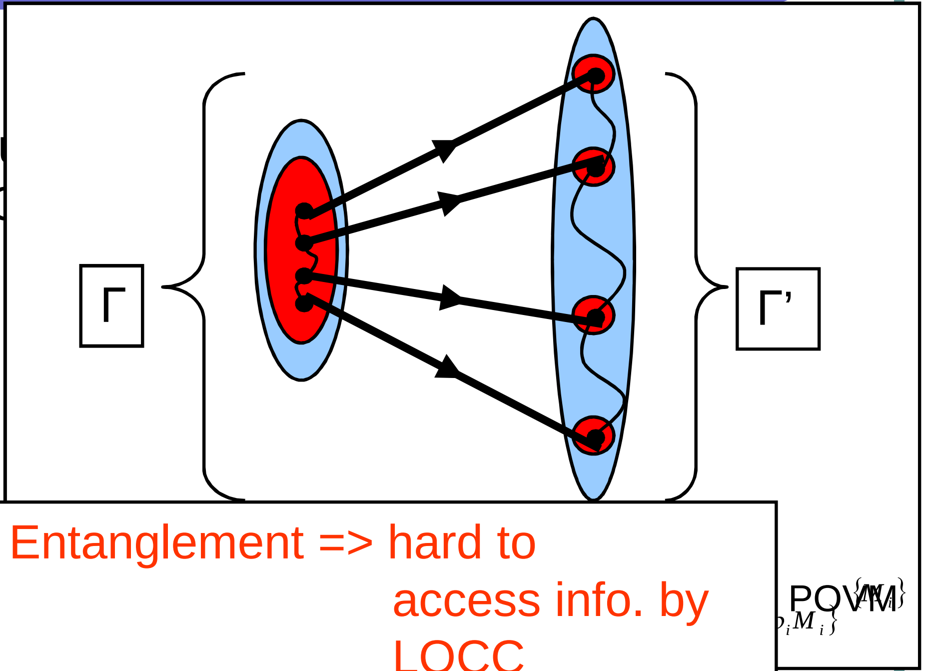

- Entanglement => hard to access info. by LOCC
- Connection allows results from both areas to be shared

Proof: channel capacity

Two steps

(1) Number of “different” messages = number of states N that can be discriminated with probability $\geq \chi$ is bounded by

$$N \leq \frac{D}{\chi \overline{2^{G}}}$$

(2) Take limits

$\Rightarrow$

$$C \leq \log_2 D - G_{\min}$$

Proof: Capacity, (1) bounds on N

To discriminate $\{\rho_i\}_{i=1..N}$, the POVM $\{M_i\}$ D = total dimension

I) $\sum_i M_i = \mathbf{I}$

II) $\mathbf{I} \geq M_i \geq 0$

POVM (*Positive Operator Valued Measurement*: mathematical description of most general measurement allowed by QM)

III) $Tr(M_i \rho_i) = 1$ Deterministic discrimination

VI)* $M_i \in SEP$ Necc. for LOCC

* Terhal, DiVincenzo, Leung, PRL 83, 5807, (2001)

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connection to states $M_i = s_i \omega_i$

$s_i \geq 0$

Density matrix

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III) $\Rightarrow s_i = \dfrac{p_i}{Tr(\omega_i \rho_i)} \geq \dfrac{1}{Tr(\omega_i \rho_i)}$

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$\Rightarrow$

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ii) $\mathbf{I} \geq s_i \omega_i \geq 0$

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$\Rightarrow$

i) $\sum_i s_i = D$

ii) $\mathbf{I} \geq s_i \omega_i \geq 0$

iii) $s_i = \frac{1}{Tr(\omega_i \rho_i)}$

iv) $\omega_i \in SEP$

Now

$$\sum_i s_i \omega_i = \mathbf{I} \Rightarrow \sum_i s_i = D$$

Proof: Capacity, (1) bounds on N

To discriminate $\{\rho_i\}_{i=1..N}$, the POVM $\{M_i\}$

D = total dimension

$$\left.\begin{array}{ll} \text{I)} & \sum_i M_i = \mathbf{I} \\ \text{II)} & \mathbf{I} \geq M_i \geq 0 \\ \text{III)} & Tr(M_i \rho_i) = 1 \\ \text{VI)} & M_i \in SEP \end{array}\right\} \Rightarrow \begin{array}{ll} \text{i)} & \sum_i s_i = D \\ \text{ii)} & \mathbf{I} \geq s_i \omega_i \geq 0 \\ \text{iii)} & s_i = \frac{1}{Tr(\omega_i \rho_i)} \\ \text{iv)} & \omega_i \in SEP \end{array}$$

Now define $d(\rho_i) := \min \frac{1}{Tr(\omega_i \rho_i)}$

$$\left.\begin{array}{l} d(\rho_i) := \min \frac{1}{Tr(\omega_i \rho_i)} \\ \text{s.t.} \quad i)1 \geq \frac{\omega_i}{Tr(\omega_i \rho_i)} \geq 0, \ ii)\omega_i \in sep \end{array}\right\}$$

By definition

$$s_i \geq d(\rho_i)$$

Proof: Capacity, (1) bounds on N

To discriminate $\{\rho_i\}_{i=1..N}$, the POVM $\{M_i\}$

D = total dimension

$$\left.\begin{array}{ll} \text{I)} & \sum_i M_i = \mathbf{I} \\ \text{II)} & \mathbf{I} \ge M_i \ge 0 \\ \text{III)} & Tr(M_i \rho_i) = 1 \\ \text{VI)} & M_i \in SEP \end{array}\right\} \Rightarrow \sum_i d(\rho_i) \le D$$

$$d(\rho_i) := \frac{1}{\max \; Tr(\omega_i \rho_i)}$$

s.t. $i)1 \ge \dfrac{\omega_i}{Tr(\omega_i \rho_i)} \ge 0$

$ii)\omega_i \in sep$

Proof: Capacity, (1) bounds on N

To discriminate $\{\rho_i\}_{i=1..N}$, the POVM $\{M_i\}$

D = total dimension

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$ii)\omega_i \in sep$

Now

$$d(\rho_i) \geq 2^{r(\rho_i)} \geq 2^{E_R(\rho_i)+S(\rho_i)} \geq 2^{G(\rho_i)}$$

Proof: Capacity, (1) bounds on N

To discriminate $\{\rho_i\}_{i=1..N}$, the POVM $\{M_i\}$

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$$N \leq \frac{D}{2^{\overline{r(\rho_i)}}} \leq \frac{D}{2^{\overline{E_R(\rho_i)+S(\rho_i)}}} \leq \frac{D}{2^{\overline{G(\rho_i)}}}$$

Now

$$d(\rho_i) \geq 2^{r(\rho_i)} \geq 2^{E_R(\rho_i)+S(\rho_i)} \geq 2^{G(\rho_i)}$$

Multiply and divide by N

Proof: Capacity, (1) bounds on N

Similarly for probabilistic discrimination

I) $\sum_i M_i = \mathbf{I}$

II) $\mathbf{I} \geq M_i \geq 0$

III) $Tr(M_i \rho_i) = q_i \geq \chi$

VI) $M_i \in SEP$

⇒

$$N \leq \frac{D}{\chi 2^{\overline{E(\rho_i)}}}$$

Proof: Capacity, (2) taking limits

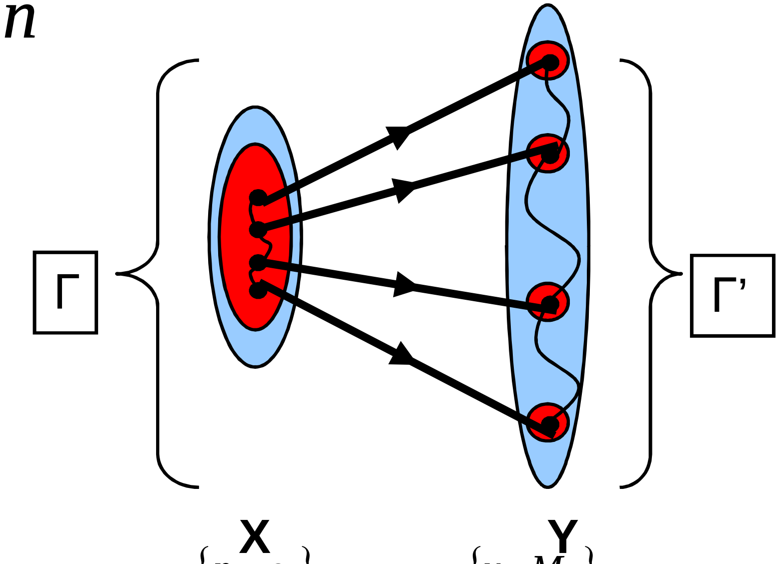
Bound…

- By using probabilistic state disc^n bound on N (with error tend to zero)

$$C \le \log D - \lim_{n\to\infty} \overline{E(\rho_\Gamma)} / n$$

- If entanglement is *additive*

$$C \le \log_2 D - E$$



$\{p_i, \rho_i\}$

$\{y_i, M_i\}$

LOCC POVM $\{M_i\}$

$y_i = p_i tr\{\rho_i M_i\}$

$\Gamma = (i_1, i_2, i_3, \ldots\ldots i_n)$

$\Updownarrow$

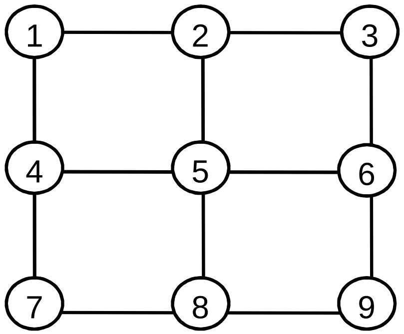
$\rho_\Gamma = \rho^1_{i_1} \otimes \rho^2_{i_2} \otimes \rho^3_{i_3} \otimes \ldots \otimes \rho^n_{i_n}$

V. Examples: 2-colourable graph states

- GRAPH STATES: Map from graphs to states

Vertex ⇔ qubit

Edge (Ed) ⇔ Entanglement (Control-Z)



n - qubits

- Set of 2^n orthonormal states $\left\{ \left| G_{k_1,..k_i,..,k_n} \right\rangle \right\}$ $k_i = 0,1$

$$\left| G_{k_1,k_2,k_3,k_4} \right\rangle = \prod_{i=1}^{n} Z_i^{k_i} \prod_{(j_1,j_2)\in Ed} CZ_{j_1,j_2} \left| + \right\rangle^{\otimes n}$$

- ‘Common eigenstates of the group generated by:’

$$K_i := X_i \otimes \prod_{(i,j)\in Ed} Z_j$$

$$K_i \left| G_{k_1,..k_i,..,k_n} \right\rangle = (-1)^{k_i} \left| G_{k_1,..k_i,..,k_n} \right\rangle \quad k_i = 0,1$$

V. Examples: 2-colourable graph states

Important Graph States in QI

• Cluster	*one-way QC*
• GHZ	*quantum crypto*
• Code States	*error correction*

All ‘Two Colourable”

Means:

Colouring into 2 colours, Red and Blue, such that

- No red connects to another red

- No blue connects to another blue

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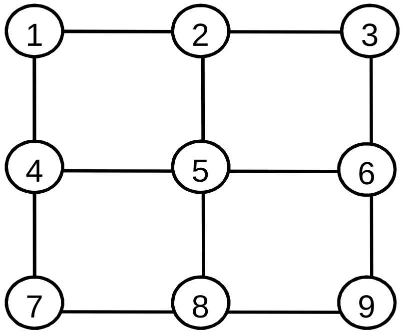

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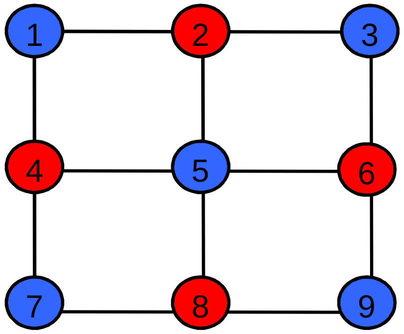

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- Use upper and lower bounds

Specific protocol $\leq N \leq \dfrac{D}{2^{\overline{E(\rho_i)}}} \leq$ Biparite entanglement bound

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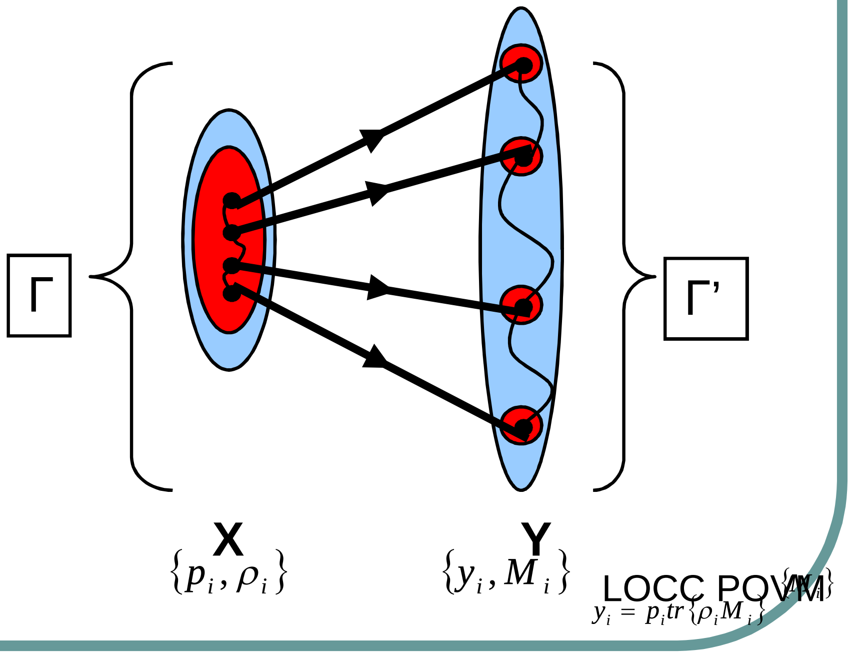
Specific protocol $\leq N \leq \frac{D}{2^{\overline{E(\rho_i)}}} \leq$ Biparite entanglement bound

- Achieved!

	N	E	C
GHZ	2^{n-1}	1	$n-1$
1D Cluster	$2^{\left\lceil \frac{n}{2} \right\rceil}$	$\left\lfloor \frac{n}{2} \right\rfloor$	$\left\lceil \frac{n}{2} \right\rceil$
2D Cluster			
3D Cluster			
Ring			
Steane code			



Discussion: usefulness of result...

Why access information by LOCC?

Could be many uses, here are a few

1) Environment(s) assisted channel capacity

    - Measures to environment of limited help

$$C_{Env.Asstd} \leq \log D - E$$

2) Distribution of secret (secret sharing/ data hiding)

    - gives bound to "secret information", but no guarantee we can use it..

$$I_{secret} := I_{global} - I_{LOCC} \geq E$$

3) Network of Quantum Computer

    - Limits amount that can be shared, if entangled

$$C_{Shared} \leq \log D - E$$

VI. Conclusions

- Entanglement → difficult to access info locally
- Use connection to aid in both directions
- Measures calculated for large sets of states
- Experiments 8-party ent.

	N	E	C
GHZ	2^{n-1}	1	$n-1$
1D Cluster	$2^{\left\lceil \frac{n}{2} \right\rceil}$	$\left\lfloor \frac{n}{2} \right\rfloor$	$\left\lceil \frac{n}{2} \right\rceil$
2D Cluster			
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Ring			
Steane code			

Future work: - uses for protocols, data hiding, error correction…

- realistic schemes

- applications to many-body physics

PRL 96 (2006), 040501, quant-ph/0606103, quant-ph/0609103 and quant-ph/soonihope