# On the equivalence and conjugacy of weighted automata

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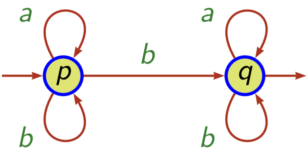
# *Part I*

## *A brief view on weighted automata*

# A first example



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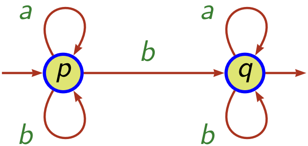


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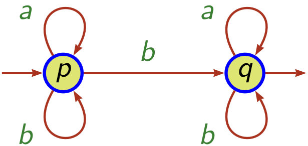


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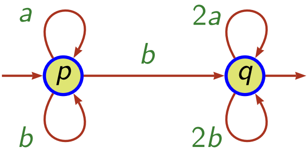

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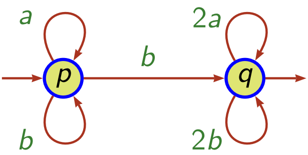


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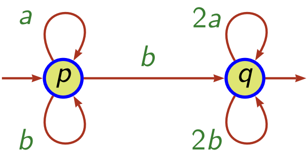


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$$\forall w \in A^* \qquad w \quad \longmapsto \quad \langle w \rangle_2$$

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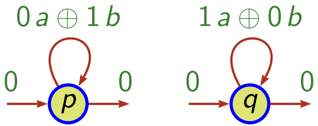

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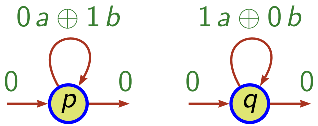


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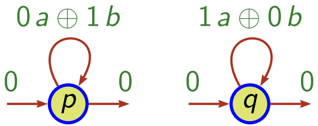


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# Universality of the multiplicity

- $\mathbb{B}$ classical automata
- $\mathbb{N}$ classical counting
- $\mathbb{Z}$ , $\mathbb{Q}$ , $\mathbb{R}$ numerical multiplicity
- $\mathcal{M} = \langle \mathbb{N}, \min, + \rangle$ Min-plus semiring
- $\mathfrak{P}(B^*) = \mathbb{B}\langle\langle B^* \rangle\rangle$ Transducers
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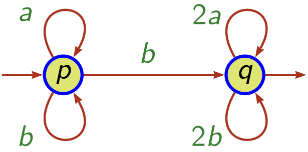
- $\mathfrak{P}(F(B))$ Pushdown automata
- $\mathrm{Rat}\, \mathbb{N}^k$ Timed automata

# The semiring of formal power series

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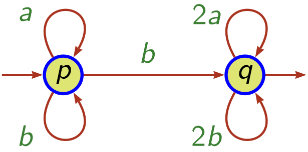


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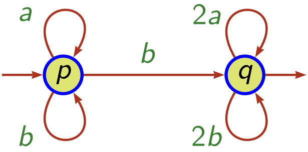


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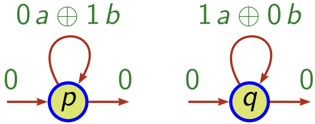


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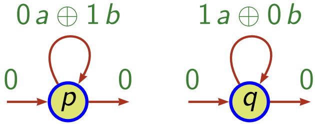


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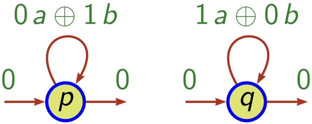


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For any semiring $\mathbb{K}$ , $\mathbb{K}^{Ae}$ is a semiring, denoted by $\mathbb{K}\langle\langle A^*\rangle\rangle$ , defined by:

| | | | |
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| addition | $s+t$ | $\forall f \in A^*$ | $\langle s+t, f\rangle = \langle s, f\rangle + \langle t, f\rangle$ , |
| (Cauchy) product | $s\,t$ | $\forall f \in A^*$ | $\langle s\,t, f\rangle = \sum_{\substack{u,v \in A^* \\ uv=f}} \langle s, u\rangle\langle t, v\rangle$ |

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a $\mathbb{K}$-semi-algebra indeed:

$$k\,s \qquad \forall f \in A^* \qquad <k\,s, f> = k<s, f>$$

$$s\,k \qquad \forall f \in A^* \qquad <s\,k, f> = <s, f>k \ ,$$

# The semiring of formal power series

**Series play the role of languages**

$\mathbb{K}\langle\langle A^* \rangle\rangle$ **plays the role of** $\mathfrak{P}(A^*)$

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Questions and problems:

- How to give a meaning to such an expression?
- For which $s$ is the star operation defined?
- What are the “properties” of the star?

# Rationality

Theorem

[**The fundamental theorem of automata**]

*What you can do with automata is exactly*

*what you can do with rational operations.*

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Equivalence may be decidable

within certain (decidable or undecidable) subfamilies

of families of automata in which equivalence is undecidable.

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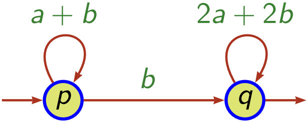
# *Part II*

# *Conjugacy and equivalence*

# Automata are matrices



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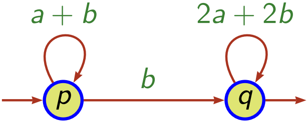


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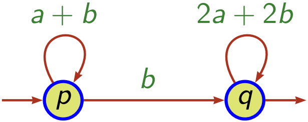


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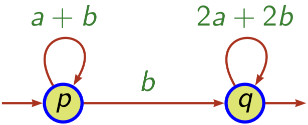


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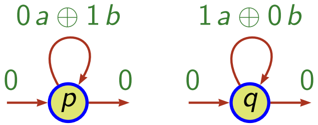

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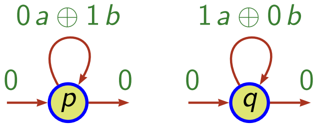


$$\mathcal{S}_1 = \left\langle \begin{pmatrix} 0 & 0 \end{pmatrix}, \begin{pmatrix} 0\,a \oplus 1\,b & +\infty \\ +\infty & 1\,a \oplus 0\,b \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\rangle$$

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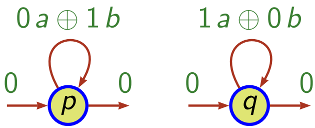


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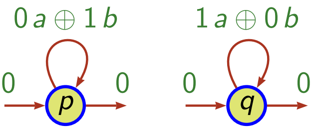


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# Conjugacy

Definition

Let $\mathcal{A} = \langle\, I, E, T \,\rangle$ and $\mathcal{B} = \langle\, J, F, U \,\rangle$ be two $\mathbb{K}$-automata.
$\mathcal{A}$ is conjugated to $\mathcal{B}$ if there exists a $\mathbb{K}$-matrix $X$ such that :

$$I\,X = J, \quad E\,X = X\,F, \quad \text{and} \quad T = X\,U \;.$$

This denoted as $\mathcal{A} \overset{X}{\Longrightarrow} \mathcal{B}$ .

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$\mathcal{A} \overset{X}{\Longrightarrow} \mathcal{B}$ implies that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*.

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Conjugacy is a *preorder*

(transitive and reflexive, but not symmetric).

$\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B}$ implies that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*.

$I\,E\,E\,T$

# Conjugacy

Definition

Let $\mathcal{A} = \langle\, I, E, T \,\rangle$ and $\mathcal{B} = \langle\, J, F, U \,\rangle$ be two $\mathbb{K}$-automata.
$\mathcal{A}$ is conjugated to $\mathcal{B}$ if there exists a $\mathbb{K}$-matrix $X$ such that :

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This denoted as $\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B}$ .

Conjugacy is a *preorder*

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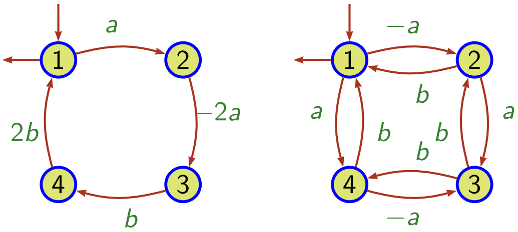
$\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B}$ implies that $\mathcal{A}$ and $\mathcal{B}$ are *equivalent*.

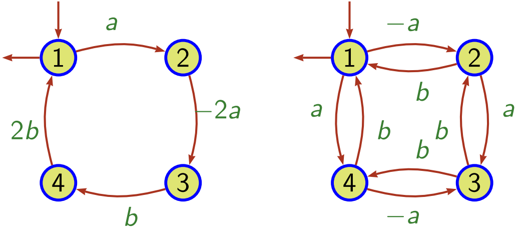
$$I\,E\,E\,T = I\,E\,E\,X\,U = I\,E\,X\,F\,U = I\,X\,F\,F\,U = J\,F\,F\,U$$

$$\text{and then} \qquad I\,E^*\,T = J\,F^*\,U$$

# Conjugacy : an example



# Conjugacy : an example




$$\begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & -2a & 0 \\ 0 & 0 & 0 & b \\ 2b & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & -a & 0 & a \\ b & 0 & a & 0 \\ 0 & b & 0 & b \\ b & 0 & -a & 0 \end{pmatrix}$$

## Conjugacy and equivalence : the result

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Theorem

*Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two* $\left\{ \begin{array}{l} \textit{Boolean automata} \\ \mathbb{N}\textit{-automata} \\ \mathbb{Z}\textit{-automata} \\ \mathbb{K}\textit{-automata, } \mathbb{K} \textit{ a field} \\ \textit{functional transducers} \end{array} \right.$ .

*If* $\mathcal{A}$ *and* $\mathcal{B}$ *are equivalent then there exists a* $\mathcal{C}$ *such that:*

$$\mathcal{A} \overset{X}{\Longleftarrow} \mathcal{C} \overset{Y}{\Longrightarrow} \mathcal{B}$$

# *Part III*

## *Conjugacy and automata morphisms*

# Automata morphisms

## Overall idea of morphisms

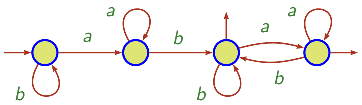
Can we do the same with less?

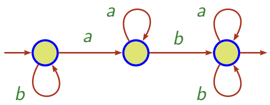
Can we do the same in another way?

Transformations that respect the *structure* of automata

# Automata morphisms

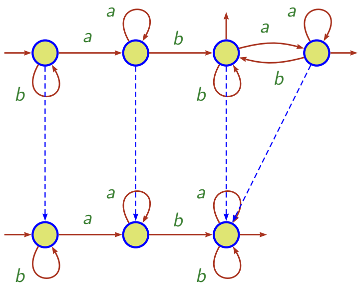
## Reduction of deterministic automata



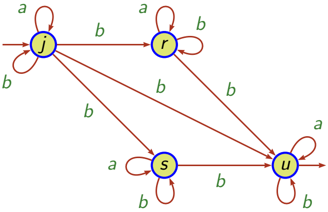



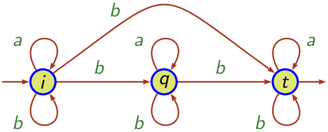
# Automata morphisms

## Reduction of deterministic automata

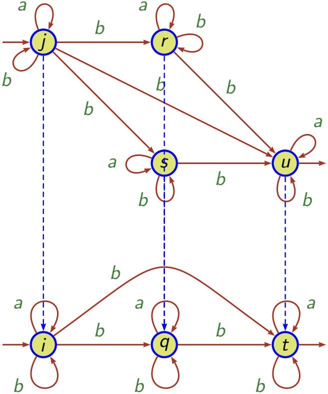


# Automata morphisms

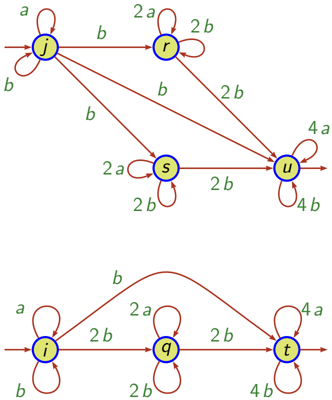


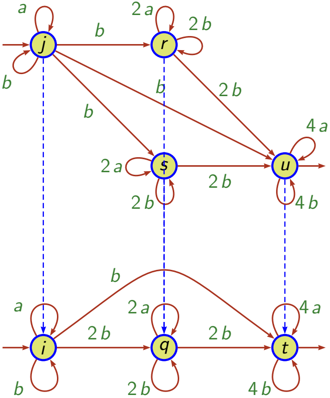
# Automata morphisms



# $\mathbb{K}$-quotient



# 𝕂-quotient



# $\mathbb{K}$-quotient

$$\underbrace{\begin{pmatrix} a+b & b & b & b \\ 0 & 2a+2b & 0 & 2b \\ 0 & 0 & 2a+2b & 2b \\ 0 & 0 & 0 & 4a+4b \end{pmatrix}}_{R_2} \left.\vphantom{\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}}\right\} R_2$$

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$$\underbrace{\begin{pmatrix} a+b & 2b & b \\ 0 & 2a+2b & 2b \\ 0 & 2a+2b & 2b \\ 0 & 0 & 4a+4b \end{pmatrix}}_{Q_2} \left.\vphantom{\begin{pmatrix} a \\ 0 \\ 0 \\ 0 \end{pmatrix}}\right\} R_2$$

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## $\mathbb{K}$-quotient is a conjugacy

$$\begin{pmatrix} a+b & b & b & b \\ 0 & 2a+2b & 0 & 2b \\ 0 & 0 & 2a+2b & 2b \\ 0 & 0 & 0 & 4a+4b \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

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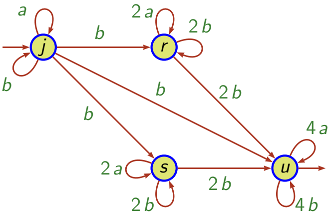
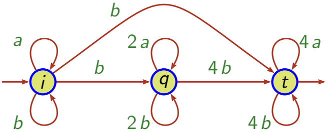
Proposition

*If* $\varphi \colon \mathcal{A} \to \mathcal{B}$ *is a* $\mathbb{K}$*-quotient, then* $\mathcal{A} \xrightarrow{H_\varphi} \mathcal{B}$ *.*

# co-$\mathbb{K}$-quotient



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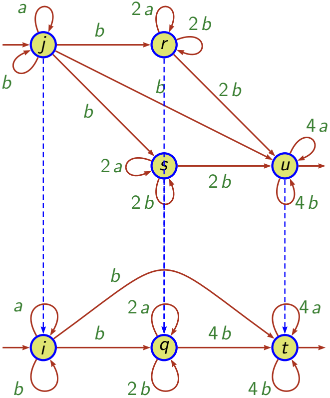

## co-$\mathbb{K}$-quotient

$$\underbrace{\begin{pmatrix} a+b & b & b & b \\ 0 & 2a+2b & 0 & 2b \\ 0 & 0 & 2a+2b & 2b \\ 0 & 0 & 0 & 4a+4b \end{pmatrix}}_{R_2} \quad \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} R_2$$

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$$\underbrace{\begin{pmatrix} a+b & b & b & b \\ 0 & 2a+2b & 2a+2b & 4b \\ 0 & 0 & 0 & 4a+4b \end{pmatrix}}_{R_2} \quad \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} Q_2$$

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# co-$\mathbb{K}$-quotient is a 'co-conjugacy'

$$\begin{pmatrix} a+b & b & b \\ 0 & 2a+2b & 4b \\ 0 & 0 & 4a+4b \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

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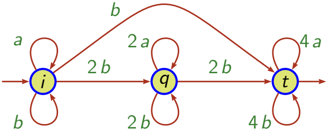
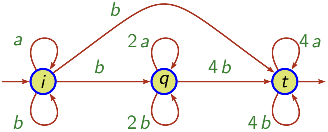
Proposition

*If* $\varphi\colon \mathcal{A} \to \mathcal{B}$ *is a co-*$\mathbb{K}$*-quotient, then* $\mathcal{B} \xRightarrow{H_\varphi^{\mathrm{t}}} \mathcal{A}$ .

# $\mathbb{K}$-circulation of coefficients



# $\mathbb{K}$-circulation of coefficients


Definition

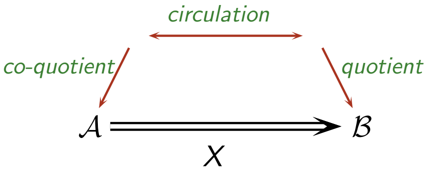
A $\mathbb{K}$-circulation of coefficients is a conjugacy by a square *diagonal* matrix of *invertible* elements of $\mathbb{K}$ .

# The result

**Theorem**

*Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two* $\begin{cases} \mathbb{Z}\text{-automata} \\ \mathbb{K}\text{-automata, } \mathbb{K} \text{ a field} \\ \text{functional transducers} \end{cases}$ .

*Then, if* $\mathcal{A} \stackrel{X}{\Longrightarrow} \mathcal{B}$ *, it holds:*

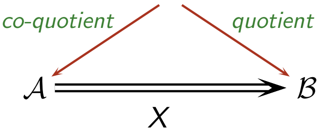


# The result

**Theorem**

*Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be two* $\begin{cases} \text{trim Boolean automata} \\ \text{trim } \mathbb{N}\text{-automata} \end{cases}$ .

*Then, if* $\mathcal{A} \overset{X}{\Longrightarrow} \mathcal{B}$ *, it holds:*



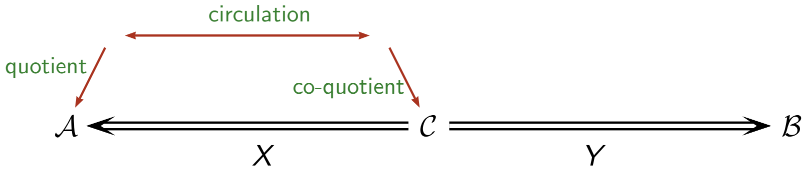
## Equivalence and $\mathbb{K}$-morphisms

The structural characterization of equivalence

$$\mathcal{A} \xLeftarrow[X]{\qquad\qquad} \mathcal{C} \xRightarrow[Y]{\qquad\qquad} \mathcal{B}$$

## Equivalence and $\mathbb{K}$-morphisms

The structural characterization of equivalence



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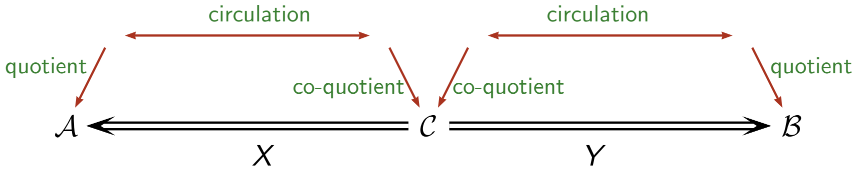
The structural characterization of equivalence



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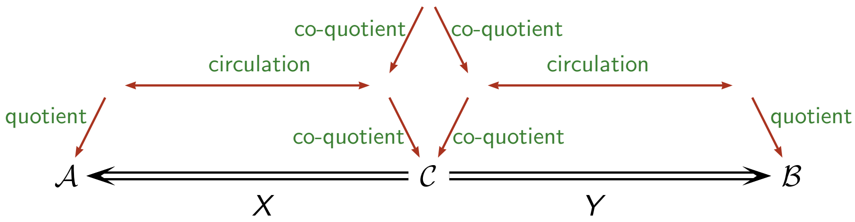
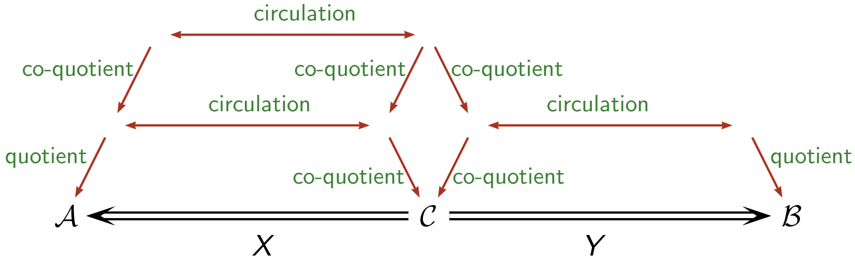
The structural characterization of equivalence



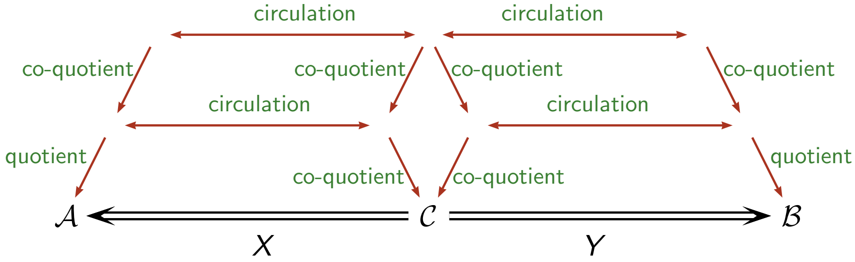
# Equivalence and $\mathbb{K}$-morphisms

## The structural characterization of equivalence



# Equivalence and $\mathbb{K}$-morphisms

## The structural characterization of equivalence



# *Part IV*

## *An application to automatic structures*

## The result

Proposition

*If two languages have the same growth function,*
*then there exists a letter-to-letter rational bijection*
*that maps one language onto the other.*

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*If two languages have the same growth function,*
*then there exists a letter-to-letter rational bijection*
*that maps one language onto the other.*

$$\forall L \subseteq A^* \,,\ \forall n \in \mathbb{N} \qquad g_L(n) \;=\; \|\{w \mid w \in L\}\|$$

## An example

$$L_1 = a\,(a+b)^* \qquad \text{and} \qquad L_2 = (c + d\,c + d\,d)^* \smallsetminus c\,c\,(c+d)^*$$

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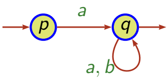
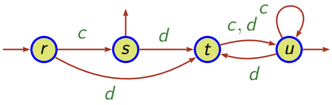
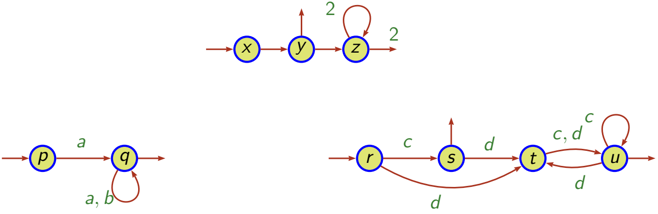
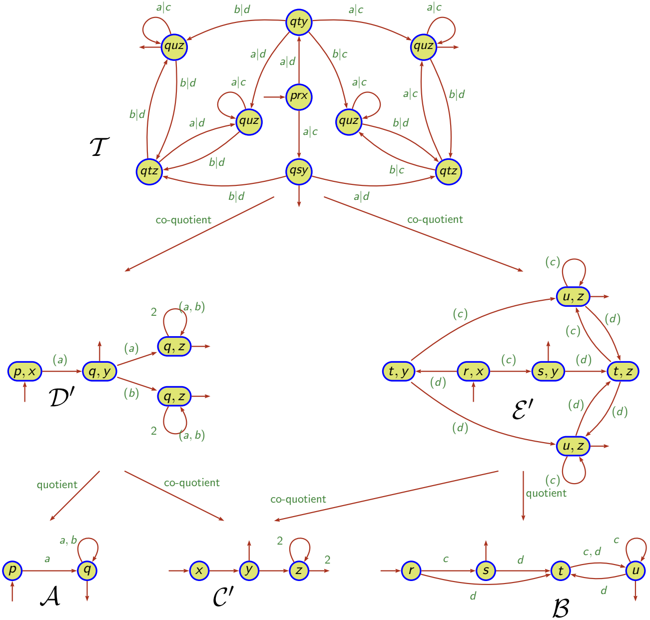


# An example

$$L_1 = a\,(a + b)^* \qquad \text{and} \qquad L_2 = (c + d\,c + d\,d)^* \smallsetminus c\,c\,(c + d)^*$$





$\mathcal{T}$
qty
prx
qsy
quz
qtz
a|c
b|d
a|d
b|c
co-quotient
co-quotient
$\mathcal{D}'$
p, x
q, y
q, z
(a)
(b)
(a, b)
2
$\mathcal{E}'$
t, y
r, x
s, y
t, z
u, z
(c)
(d)
quotient
co-quotient
co-quotient
quotient
$\mathcal{A}$
p
q
a
a, b
$\mathcal{C}'$
x
y
z
2
$\mathcal{B}$
r
s
t
u
c
d
c, d

# Conclusion: open questions

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The relationship between

*conjugacy*, *equivalence*, and $\mathbb{K}$-*morphisms*

remains open for several semirings:

min-plus or max-plus semirings, non functional transducers,

...

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The relationship between

*conjugacy*, *equivalence*, and $\mathbb{K}$-*morphisms*

remains open for several semirings:

min-plus or max-plus semirings, non functional transducers,

...

Are there cases

where equivalence is undecidable and conjugacy *decidable* ?